CONCERNING WEBS IN THE PLANE

BY

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1. **Introduction.** A compact continuum M is said[1] to be a web if and only if there exist two upper semicontinuous collections G_1 and G_2 such that (1) each of these collections fills up M, (2) each of them is a dendron with respect to its elements, and (3) there exists an uncountable subcollection W of the collection G_1 such that no element of W is a subset of any element of G_2. If there exist two collections G_1 and G_2 satisfying all of the above conditions together with the additional condition that every element of G_1 intersects each element of G_2 in a totally disconnected point set, then the web M is said[2] to be simple.

In the above definition of a web "dendron" may be replaced[3] by "arc." But R. H. Bing has shown[4] by an example that a corresponding change in the definition of *simple* web would alter the meaning of that term.

In the present paper the notions of a W_n set and a W_n' set are introduced.

DEFINITION. *If $n>1$, a W_n set is a compact continuum M for which there exists a family F of n elements such that* (1) *each element of F is an upper semicontinuous collection of mutually exclusive continua which fills up M and is an arc with respect to its elements,* (2) *if G is a collection of continua each belonging to some, but no two to the same, collection of the family F, then the continua of the collection G have a point in common and their common part is totally disconnected.*

DEFINITION. *A W_n' set is a W_n set satisfying the conditions obtained by replacing, in the above definition, the phrase "upper semicontinuous collection of mutually exclusive continua" by the phrase "continuous collection of mutually exclusive continuous curves."*

Hereafter in this paper the space considered will be the Euclidean plane. It will be shown that there exists a W_2 set which is not a W_3 set and that for each positive integer n a simple closed curve plus its interior is a W_n set. A necessary and sufficient condition for a compact continuum to be a W_2' set will also be established. I wish to express my deep appreciation to Professor

Presented to the Society, September 5, 1952, the theorems of §7 were presented under the title *Concerning a certain type of web*; received by the editors June 9, 1952 and, in revised form, August 11, 1952.

(1) R. L. Moore, *Concerning continua which have dendratomic subsets*, Proc. Nat. Acad. Sci. U. S. A. vol. 29 (1943) pp. 384–389.

(2) R. L. Moore, *A characterization of a simple plane web*, Proc. Nat. Acad. Sci. U. S. A. vol. 32 (1946) pp. 311–316; and R. H. Bing, *Concerning simple plane webs*, Trans. Amer. Math. Soc. vol. 60 (1946) pp. 133–148.

(3) Cf. Theorem 1 of paper cited in footnote 1.

(4) R. H. Bing, loc. cit.

R. L. Moore who aroused my interest in these problems and whose many helpful suggestions were an invaluable aid in the preparation of this paper.

2. Filling up a compact continuous curve whose boundary is the sum of two mutually exclusive simple closed curves.

THEOREM 1. *If M is the compact continuum whose boundary is the sum of two circles C_1 and C_2 with center at the origin and radii* 1 *and* 2 *respectively, then there exists a family F of n elements such that* (1) *each element of F is a continuous collection of mutually exclusive arcs which fills up M and is a simple closed curve with respect to its elements,* (2) *if G is a collection of arcs each belonging to some, but no two to the same, collection of the family F, then the arcs of the collection G have a point in common and their common part is totally disconnected.*

Proof. We shall begin by proving the theorem in detail for the case $n=3$. Let S be the square plus its interior with vertices (0, 0), (0, 1), (1, 0), and (1, 1). Hereafter, when referring to S we shall use rectangular co-ordinates; when referring to M, polar co-ordinates. Let I be an interval of length 1 with end points A and B.

For each non-negative number a not greater than 1, let m_a be the collection of all points in S with ordinate a and let k_a be the collection of all those with abscissa a. For each non-negative number a not greater than 2, let r_a be the collection of all points (r, θ) in M such that $\theta = 2\pi a$.

There exists a continuous transformation T throwing I into S such that if the subset K of I contains a domain with respect to I, $T(K)$ contains a domain with respect to S and $T(A)$ and $T(B)$ are vertices of S.

For each non-negative number z less than 1 let T_{1z} denote a transformation of I into a subset I_{1z} of M such that if P is a point of I such that $AP=x$ and if $T(P)$ is a point of k_y, then $T_{1z}(P)=(1+x,\ 2\pi(y+z))$. In M, $T_{1z}(P)$ is a point of the interval r_{y+z} at a distance x from C_1.

For each z, T_{1z} is reversibly continuous and therefore I_{1z} is an arc. For each point P of M there is only one number z such that P belongs to I_{1z}. Let G_1 be the collection of all I_{1z}'s. Since, for each z, I_{1z} can be obtained from I_{10} by means of a rotation about the origin, the collection G_1 is continuous. So it is a continuous collection of mutually exclusive arcs, it fills up M, and it is a simple closed curve with respect to its elements.

If z and z' are non-negative numbers less than 1 and C is the set of all points P of I such that $T_{1z}(P)$ is a point of $r_{z'}$, then, depending on whether or not $z'-z$ is non-negative, $T(C)$ is either $k_{z'-z}$ or $k_{z'-z+1}$ and is therefore uncountable. But no interval contains a domain. Therefore C contains no segment and consequently the common part of I_{1z} and $r_{z'}$ is uncountable and totally disconnected.

Now, for each non-negative number z' less than 1 let $T_{2z'}$ denote a transformation of I into a subset $I_{2z'}$ of M such that if P' is a point of I such that $AP'=x'$ and $T(P')$ belongs to $m_{y'}$, then $T_{2z'}(P')$ is a point of the interval $r_{y'+z'}$ at a distance x' from C_1. Let G_2 be the collection of all $I_{2z'}$'s. Then G_2 is a continuous collection of mutually exclusive arcs filling up M and it is a simple closed curve with respect to its elements. Also, if z and z' are non-negative numbers less than 1, the common part of $I_{2z'}$ and r_z exists and is totally disconnected.

Let z, z', and y each denote a non-negative number less than 1. We shall prove that I_{1z}, $I_{2z'}$, and r_y have a common part. Let P' be the point of intersection of k_{y-z} (or k_{y-z+1} if $y<z$) and $m_{y-z'}$ (or $m_{y-z'+1}$ if $y<z'$), and let P be a point of I such that $T(P)=P'$. If x is the distance from P to A, $T_{1z}(P)$ is on r_y at a distance x from C_1 and $T_{2z'}(P)$ is on r_y at a distance x from C_1. Therefore $T_{1z}(P)=T_{2z'}(P)$.

We now prove that the common part of I_{1z} and $I_{2z'}$ is totally disconnected(5). Let α be an arc in M with end points (r_1, θ_1) and (r_2, θ_2) where $r_1<r_2$. Since no subinterval of I goes into an arc under T, there is a point P of I such that $r_1<AP+1<r_2$ and $[z+\text{abscissa } T(P)]-[z'+\text{ordinate } T(P)]$ is not an integer. Consequently $T_{1z}(P)$ and $T_{2z'}(P)$ do not have the same θ co-ordinate and the point of α at distance AP from C_1 is not a point of the common part of I_{1z} and $I_{2z'}$. Since this common part does not contain an arc, it is totally disconnected.

Let G_3 be the collection of all r_y's. The collections G_1, G_2, and G_3 satisfy the conditions of Theorem 1.

If k is any positive integer, the preceding arguments may be easily extended to prove Theorem 1 for the case $n=k$. We need only let S be the $(k-1)$-dimensional cube consisting of all points whose rectangular coordinates are non-negative numbers not greater than 1 and consider, in place of the intervals k_a and m_a, the intersections of S and an appropriate selection of $k-1$ planes.

It is also possible to obtain a family of infinitely many elements which satisfies the conditions of Theorem 1. Let S be the collection of all points which are infinite simple sequences each element of which is a non-negative number not greater than 1. The sequence $P_1, P_2, \cdots$ is said to converge to the point P if and only if for each positive integer n the nth elements of the sequences $P_1, P_2, \cdots$ converge to the nth element of P. The set S is seen to be a compact continuous curve and therefore the preceding arguments may be extended to this case.

3. Filling up a simple closed curve plus its interior.

THEOREM 2. *If M' is a simple closed curve plus its interior and n is a positive integer, then M' is a W_n set.*

Proof. Let C_1, C_2, and M be as defined in the proof of Theorem 1. It fol-

(5) The author is indebted to the referee, who suggested ways of shortening the argument here and of shortening the arguments in the proofs of Theorems 2, 11, and 12.

lows from the proof of Theorem 1 that if n is a positive integer, there exist n collections $G_1, G_2, \cdots, G_n$ each filling up M and satisfying the conditions of Theorem 1 such that for each i each arc of G_i has one end point on C_1 and one end point on C_2, these points having the same θ co-ordinate, and each point of C_1+C_2 is an end point of some arc of G_i.

There is an upper semicontinuous collection G of mutually exclusive point sets filling up M such that each element of G is either a point of $M-C_1$ or the common part of C_1 and some vertical line. For each i and each nonnegative number x not greater than π let h_{ix} denote the subcollection of G the sum of whose elements is the sum of the elements of G_i which contain either $(1, x)$ or $(1, -x)$. The collection h_{ix} is an arc with respect to its elements, and H_i, the collection of all h_{ix}'s, is an arc with respect to its elements. The family $H_1, H_2, \cdots, H_n$ satisfies the conditions of Theorem 2 with respect to the elements of the collection G which, with respect to its elements, is topologically equivalent to M'.

It follows from the proof of Theorem 2 and the concluding remarks in the proof of Theorem 1 that there exists a family of infinitely many collections which satisfies conditions similar to those required for a W_n set with respect to M'.

4. W_1 sets and their boundaries.

DEFINITION. *A W_1 set is a compact continuous curve M such that* (1) *every complementary domain of M is bounded by a simple closed curve which has no more than one point in common with the boundary of any other complementary domain of M, and* (2) *there exists an upper semicontinuous collection of mutually exclusive continua which fills up M and is an arc with respect to its elements.*

Throughout this paper E will denote the set of all points in the plane. If K is a subcontinuum of the W_1 set M, the notation M_K will be used to denote a point set such that P is a point of it if and only if P is a point either of K or of some complementary domain of M whose boundary is a subset of K.

The following theorem holds true. It is clear that it and Theorem 4 remain true under much less restrictive hypotheses.

THEOREM 3. *If K is a subcontinuum of the W_1 set M, then if M_K separates E, K separates both E and M and if $M-K$ is the sum of two mutually separated connected point sets L and N and M_K separates E, then $E-M_K$ is the sum of two mutually separated connected point sets of which one contains L and the other contains N.*

THEOREM 4. *If the subcontinuum K of the W_1 set M separates M and M_K does not separate E, then the common part of K and the boundary of some complementary domain of M exists and is not connected.*

Proof. There exists a complementary domain of M whose boundary contains a point of $M-K$. Otherwise $M-K$ would be identical with $E-M_K$

contrary to the supposition that K separates M and M_K does not separate E.

Let $D_1, D_2, \cdots$, be the complementary domains of M whose boundaries intersect $M-K$. Let $J_1, J_2, \cdots$, be their respective boundaries. By hypothesis $M-K$ is the sum of two mutually separated point sets L and N. Suppose that for each i either J_i does not contain a point of K or $K\cdot J_i$, and consequently $J_i-K\cdot J_i$, is connected. Let K_i be the set of all points of J_i that do not belong to K. For each i, K_i is connected and is therefore a subset either of L or of N. Let N_D be the sum of all the D_i's such that K_i is a subset of N and let L_D be the sum of all others. The point set $E-M_K$ is the sum of the mutually separated point sets $L+L_D$ and $N+N_D$, contrary to the hypothesis.

Suppose M is a W_1 set and G is an upper semicontinuous collection of mutually exclusive continua which fills up M and is an arc with respect to its elements. Then the following Theorems 5–9 hold true.

THEOREM 5. *If g is an end element of G, then there do not exist two complementary domains of M each bounded by a simple closed curve which intersects g but does not lie wholly in it.*

Proof. Suppose there do exist two such complementary domains D_1 and D_2 with boundaries J_1 and J_2 respectively. Since the continuum g does not separate M, $J_1\cdot g$ and $J_2\cdot g$ are connected. Let t_1 and t_2 be arcs each of which lies except for one end point in $J_1-J_1\cdot g$ and does not contain a point of the common part of the other one and $J_1-J_1\cdot g$. Let t_3 and t_4 be two such arcs in J_2.

For each i let H_i denote the collection of all elements of G which intersect t_i. The collection H_i is an arc with respect to its elements and g is an end element of this arc. Since g is also an end element of G, the common part of the collections H_1, H_2, H_3, and H_4 is a collection H' which is an arc with respect to its elements. Every element of H' other than g intersects each $t_i-t_i\cdot g$. Moreover, there is an element, h, of H' distinct from g which does not contain any point of J_1+J_2 not in one of the t_i's. Neither of the point sets $h\cdot J_1$ and $h\cdot J_2$ is connected. Consequently there are points P_1 and P_2 of J_1 and J_2 respectively such that h separates g from P_1 in $E-D_1$ and g from P_2 in $E-D_2$. But D_2 is a subset of $E-D_1$ and since g intersects J_2, h separates $g+P_2$ from P_1 in $E-D_1$. But h, g, P_1, and P_2 are subsets of $E-(D_1+D_2)$ and M. Consequently h separates g from P_1, P_1 from P_2, and P_2 from g in M, which is impossible, since $M-h$ is the sum of two mutually separated *connected* point sets. Therefore the supposition that there exist domains such as D_1 and D_2 is false.

THEOREM 6. *Suppose g is an element of G which separates M. If M_g does not separate E there do not exist two complementary domains of M each bounded by a simple closed curve which intersects g but does not lie wholly in it, and if M_g does separate E there do not exist three such domains.*

Proof. If M_g does not separate E, then by Theorem 4 there exists a simple closed curve J_1 bounding some complementary domain of M and such that $J_1 \cdot g$ exists and is not connected. Suppose there exists a simple closed curve J_2, different from J_1, which is the boundary of a complementary domain of M and intersects, but is not a subset of, g. The common part of J_2 and g is connected. Otherwise it would follow from the proof of Theorem 5 that $M-g$ is the sum of three mutually separated point sets, which is impossible.

The point set $M-g$ is the sum of two mutually separated connected point sets L and N. The point set $J_2-J_2 \cdot g$ is connected and therefore is a subset either of L or of N. Suppose it is a subset of L. Since $J_1 \cdot g$ is not connected, there exist two points of J_1 which are separated in M by g and since L and N are connected one of these points belongs to L and the other to N. Therefore there exists an arc t on J_1 which lies wholly in L except for its end points, which belong to g, and there exist two nonintersecting arcs t_1 and t_2 each of which is a subset of t and has an end point in g. Also, there exist two arcs t_3 and t_4 on J_2 such that each of them lies except for one end point in $J_2-J_2 \cdot g$ and contains no point of the other one not in g. The argument of Theorem 5 may now be used to prove that $M-g$ is the sum of three mutually separated point sets. But this involves a contradiction.

Suppose M_g does separate E and that there exist *three* simple closed curves J_1, J_2, and J_3, each of which bounds a complementary domain of M and intersects g but is not a subset of it. For each i let t_i be an arc on J_i which lies except for its end points in $J_i-J_i \cdot g$, or, if $J_i \cdot g$ consists of only one point, let t_i be J_i. Each of the point sets $t_1-t_1 \cdot g$, $t_2-t_2 \cdot g$, and $t_3-t_3 \cdot g$ is connected and is a subset either of L or of N. Two of these are either both subsets of L or both subsets of N. The argument of Theorem 5 may now be used to prove that L is not connected. This is a contradiction.

THEOREM 7. *If g is an element of G and M_g separates E, then* (1) *if J_1 is the boundary of a complementary domain of M and intersects g, their common part is connected and* (2) *if J_1 is not a subset of g and J_2 is the boundary of a complementary domain of M, is different from J_1, and intersects g but is not a subset of it, then $J_1-J_1 \cdot g$ and $J_2-J_2 \cdot g$ are separated in M by g.*

Proof. It follows from Theorem 3 that $M-g$ is the sum of two mutually separated connected point sets L and N. It also follows from Theorem 3 that $E-M_g$ is the sum of two mutually separated connected point sets L' and N' containing L and N respectively.

Let D_1 be the complementary domain of M whose boundary is J_1 and suppose that $J_1 \cdot g$ exists and is not connected. Since J_1 is not a subset of g, $E-M_g$ contains D_1 and D_1 is a subset either of L' or N'—say L'. Consequently $J_1-J_1 \cdot g$ is a subset of L' and of L, contrary to the fact that g separates two points of J_1 in M. Therefore $J_1 \cdot g$ is connected.

If neither of J_1 and J_2 is a subset of g, the point sets $J_1 - J_1 \cdot g$ and $J_2 - J_2 \cdot g$ are connected. It follows from the proof of Theorem 6 that g separates these point sets in M and from Theorem 3 that M_g separates them in E. Consequently g separates $J_1 - J_1 \cdot g$ from $J_2 - J_2 \cdot g$ in E.

THEOREM 8. *Suppose J, the boundary of some complementary domain of M, is not a subset of any element of G. Let H be the collection of all elements of G that intersect J. Then* (1) *there exist two elements h and h' of H such that $h \cdot J$ and $h' \cdot J$ are connected and* (2) *if g is an element of H other than h or h', M_g does not separate E and $J \cdot g$ is not connected.*

Proof. The collection H is an arc with respect to its elements. Let h and h' denote its end elements. Suppose $h \cdot J$ is not connected. There exist two points A and B of J that are separated by h in M. The elements of H that contain A and B are separated by h in M and consequently in H^*(6). This is impossible. Hence $J \cdot h$ and $J \cdot h'$ are connected.

By Theorems 3, 4, and 6, if h is not an end element of G (and consequently separates M), then h separates E.

Let g be an element of H other than h or h'. The point set g separates h from h' in M. Therefore $J \cdot g$ separates $J \cdot h$ from $J \cdot h'$ in J, which is possible only if $J \cdot g$ is not connected. It follows from Theorem 7 that M_g does not separate E.

COROLLARY. *If J, the boundary of some complementary domain of M, is not a subset of any element of G, H denotes the collection of all elements of G that intersect J, h and h' denote the end elements of H, and J' is the boundary of a complementary domain of M, is different from J, and intersects $H^* - (h + h')$, then J' is a subset of some element of H.*

Proof. The simple closed curve, J', intersects some element g of H different from h or h'. The point set $J \cdot g$ is not connected, so M_g does not separate E. It follows from Theorem 6 that J' is a subset of g.

THEOREM 9. *Let α be a sequence of boundaries of distinct complementary domains of M no term of which is a subset of any element of G. If the limiting set of α is a nondegenerate continuum, it is a subset of some element of G.*

Proof. Let K denote the limiting set of α. Suppose K intersects two elements, g_1 and g_2, of G. Let S denote the sum of all elements of G between g_1 and g_2. Since K is connected, it intersects S and consequently some element, J, of α intersects S. Since J is not a subset of any element of G, it follows from Theorem 8 that there is an element, g, of G between g_1 and g_2 such that $J \cdot g$ exists and is not connected. The point set $M - g$ is the sum of two mutually separated connected point sets L and N. There exist two points of J

(6) If H is a collection of mutually exclusive point sets the notation H^* is used to denote the sum of the elements of H.

that are separated by g in M. Hence L and N both intersect J. The point sets $J \cdot L$ and $J \cdot N$ are both connected, for otherwise g would separate M into three mutually separated point sets.

Suppose K is not a subset of g. Then it intersects either L or N—say L. If H_1 denotes the collection of all elements of G that intersect $L+g$, H_1 is an arc with respect to its elements and g is an end element of this arc. If H_2 denotes g plus the collection of all elements of G that intersect $J \cdot L$, H_2 is an arc with respect to its elements, g is an end element of this arc, and H_2 is a subcollection of H_1. Since $H_2^* \cdot J$ is connected, there is an element h of H_2 such that $h \cdot J$ is connected and h is an end element of H_2. Let S' denote the sum of all the elements of H_2 between g and h. Since K is connected, it intersects S'. Hence, since K is the limiting set of α, there is an element J' of α which intersects $S' - S' \cdot J$. There is an element, h', of H_2 between h and g which intersects J'. But, by Theorem 8, $J \cdot h'$ is not connected and $M_{h'}$ does not separate E. Therefore, by Theorem 6, J' is a subset of h', contrary to hypothesis. Hence K does not intersect two elements of G and is therefore a subset of some element of G.

5. Boundaries of W_3' sets.

THEOREM 10. *The limiting set of any infinite sequence of distinct boundaries of complementary domains of a W_3 set is totally disconnected.*

Proof. Suppose α is an infinite sequence of distinct boundaries of complementary domains of the W_3 set M. The continuum M is a simple web and therefore a continuous curve(7). Bing has shown(8) that each complementary domain of a simple web is bounded by a simple closed curve and that no two such boundaries have more than one point in common. Consequently M is a W_1 set.

Let G_1, G_2, and G_3 be collections satisfying with respect to M the requirements of the definition of a W_3 set. Suppose K, the limiting set of α, contains a nondegenerate continuum L. The continuum L is the limiting set of a subsequence α' of α.

Suppose that there exist infinitely many terms of α' each lying in some element of G_1. Let α_1 denote the subsequence of α' whose terms are the terms of α' having this property. Suppose that the limiting set of α_1 contains a nondegenerate continuum K'. The continuum K' is the limiting set of some subsequence α_1' of α_1. No element of G_2 or G_3 contains an element of α_1'. Therefore, by Theorem 9, K' is a subset of an element of G_2 and of an element of G_3. Since K' is nondegenerate and connected, this is impossible.

Similarly there are not infinitely many elements of α' each lying in some element of G_2 or G_3, and such that the limiting set of a sequence whose terms

(7) R. L. Moore, *Concerning webs in the plane*, Proc. Nat. Acad. Sci. U. S. A. vol. 29 (1943) pp. 389–393.

(8) R. H. Bing, loc. cit., Theorem 2.

are these elements contains a nondegenerate continuum. Hence the continuum L is the limiting set of the subsequence of α' whose terms are the terms of α' that do not lie in any element of G_1, G_2, or G_3. But, since L is not a subset of an element of G_1, an element of G_2, and an element of G_3, this is contrary to Theorem 9.

6. Examples of a W_2 set and a W_3 set.

THEOREM 11. *Every compact continuum whose boundary is the sum of two mutually exclusive simple closed curves is a W_n set, for each n.*

Proof. It follows from the proof of Theorem 2 that if M' denotes C_2 plus its interior, then there exist n collections of arcs, $G_1, G_2, \cdots, G_n$, satisfying with respect to M' the conditions of Theorem 2 such that, for each i, (1) if g is an end element of G_i, $g \cdot C_2$ contains an end point of g and is either the point (2, 0) or the point (2, π) in the polar coordinate system and (2) if g is a non-end element of G, $g \cdot C_2$ is the sum of the end points of g and is the common part of C_2 and some vertical line.

Let C be an arc on C_2 which contains neither of the points (2, 0) and (2, π) and let C' be its reflection in the x-axis. There is an upper semicontinuous collection G of mutually exclusive point sets filling up M' such that each element of G is either a point of $M'-(C+C')$ or the common part of $C+C'$ and some vertical line. With respect to its elements G is topologically equivalent to the continuum M of Theorem 1, and arguments similar to those used in the proof of Theorem 2 may now be used to show that with respect to its elements G is a W_n set and that M is a W_n set. Every compact continuum whose boundary is the sum of two mutually exclusive simple closed curves is topologically equivalent to M. Hence every such continuum is a W_n set.

THEOREM 12. *There exists a W_2 set which is not a W_3 set.*

Proof. Let s_0 denote the interval [0, 1] on the x-axis. Let α denote a sequence whose terms, $p_1, p_2, \cdots$, are the rational numbers between 0 and 1. Let $s_1, s_2, \cdots$ and $m_1, m_2, \cdots$ be two infinite sequences of straight line intervals converging to s_0 such that, for each i, s_i and m_i are above the x-axis and parallel to it, each of them has one end point on the line $x=0$ and one on the line $x=1$, and s_i is between m_i and m_{i+1}.

Let $M_1, M_2, \cdots$ be an infinite sequence of squares with interiors $I_1, I_2, \cdots$ respectively such that, for each i, (1) the diagonals of M_i are parallel to the axes, (2) the center of M_i is on s_i and has abscissa p_i, and (3) M_i does not intersect m_i, m_{i+1} or either of the lines $x=0$ or $x=1$ and if $j\neq i$ it does not intersect M_j or the line $x=p_j$. Let t_0 and t_1 denote the intervals of the lines $x=0$ and $x=1$ whose end points lie on s_0 and m_1. Let N denote the rectangular disc bounded by $s_0+m_1+t_0+t_1$.

Let K be an upper semicontinuous collection of mutually exclusive continuous curves which fills up $N-\sum_{i=1}^{\infty} I_i$, is an arc with respect to its elements, and is such that its end elements are t_0 and t_1, no element of it contains a horizontal interval, and, if k is an element of K, $k\cdot(s_0+m_1)$ is the common part of s_0+m_1 and some vertical line. Let H be the collection of all point sets h such that h is the common part of $N-\sum_{i=1}^{\infty} I_i$ and some horizontal line.

Let G be an upper semicontinuous collection of mutually exclusive point sets filling up $N-\sum_{i=1}^{\infty} I_i$ such that each of its elements is either the common part of t_0+t_1 and some horizontal line between s_0 and m_1 or a point of $N-(\sum_{i=1}^{\infty} I_i+t_0+t_1)$. With respect to its elements G is topologically equivalent to a bounded subset of the plane and the methods of Theorems 2 and 11 can be used to prove that G is a W_2 set with respect to its elements. However, it follows from Theorem 10 that it is not a W_3 set.

The following theorems also hold true.

THEOREM 13. *If M is a W_3 set, B is its boundary, and J is the boundary of a complementary domain of M, then there do not exist seven points of J each of which is a limit point of $B-J$.*

THEOREM 14. *There exists a W_3 set M with boundary B and a complementary domain of M whose boundary J contains six limit points of $B-J$.*

THEOREM 15. *If M is a W_7 set, B is its boundary, and J is the boundary of a complementary domain of M, then there do not exist three points of J each of which is a limit point of $B-J$.*

7. **Concerning W_2' sets.** In what follows, if M is a compact continuum in the plane, $B(M)$ will denote its boundary and $S(M)$ will denote the collection consisting of all boundaries of complementary domains of M, all points of $B(M)$ that are not on the boundary of any complementary domain of M, and all points of $B(M)$ that are common to the boundaries of two or more complementary domains of M. If $S(M)$ is an upper semicontinuous collection of type two, let $Q_{S(M)}$ denote a space whose "points" are the elements of $S(M)$ and whose "regions" are the domains(9) of elements of $S(M)$, the "point" x being "contiguous" to the "point" y if and only if either x is an ordinary point of the continuum y or y is an ordinary point of the continuum x.

The following theorem will be proved.

THEOREM 16. *The compact continuum M in the plane is a W' set if and only if* (1) *every boundary of a complementary domain of M is a simple closed curve and* (2) *the collection $S(M)$ is an upper semicontinuous collection of type* 2 *and there exists an upper semicontinuous collection, S', of type* 2 *such that $S(M)$ is a subcollection of S', each element of $S'-S(M)$ is a point of $M-B(M)$, S' is an*

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(9) R. L. Moore, *Fundamental theorems concerning point sets*, The Rice Institute Pamphlet, vol. 23, 1936, pp. 43 and 56.

arc[10] *with respect to its elements, and if J is the boundary of a complementary domain of M there are only two points P of J such that P is a limit point of the sum of the elements of S′ different from J.*

Before proceeding to the proof of this theorem, certain lemmas will be established.

LEMMA 1. *Suppose* (1) *M is a compact continuous curve in the plane such that every boundary of a complementary domain of M is a simple closed curve,* (2) *G is a continuous collection of mutually exclusive continuous curves which fills up M and is an arc with respect to its elements, and* (3) *g is a non-end element of G which intersects the boundary J of some complementary domain of M. Then $J \cdot g$ does not contain an arc.*

Proof. Suppose $J \cdot g$ does contain an arc, t. The point set $M-g$ is the sum of two mutually separated point sets K and L. Since G is continuous, g is a subset of the common part of the boundaries of the continuous curves $K+g$ and $L+g$.

Let P_1 and P_2 be two cut points of t. There are two connected domains with respect to $K+g$, D_1 and D_2, containing P_1 and P_2 respectively such that $\overline{D}_1$ and $\overline{D}_2$ are mutually exclusive and their sum does not contain either end point of t. There is an element, g', of G, different from g, but intersecting both D_1 and D_2. Some element of G distinct from g contains an arc, t', with end points in D_1 and D_2 respectively. In D_1 there is an arc containing P_1 and some point of t' and in D_2 there is an arc containing P_2 and some point of t'. Therefore there is an arc $P_1'P_2'$ lying in K except for its end points, which are cut points of t.

Similarly there is an arc $Q_1'Q_2'$ lying in L except for its end points, which are separated in M by $P_1'P_2'$. Since K and L are mutually separated, this is impossible. Hence $J \cdot g$ does not contain an arc.

LEMMA 2. *If J is the boundary of a complementary domain of the W_2' set M, J does not contain three limit points of $B(M)-J$.*

Proof. Let G_1 and G_2 denote two collections with respect to which M is a W_2' set such that no element of G_1 contains J. The collection H of all elements of G_1 that intersect J is an arc with respect to its elements. Let h_1 and h_2 be its end elements and let H_1 denote the collection $H-(h_1+h_2)$. Suppose three points, P_1, P_2, and P_3, of J are limit points of $B(M)-J$. If one of these were a point of H_1^*, there would exist a complementary domain of M whose boundary is a subset of H_1^* and consequently a subset of some element of H_1. (See Theorem 8, Corollary.) But, since no element of H_1 is an end element of G_1, this is contrary to Lemma 1. Therefore two of the points P_1, P_2, and P_3—say P_1 and P_2—are in the same one of the sets h_1 and h_2—say h_1. It follows

(10) R. L. Moore, loc. cit., p. 2.

that $J \cdot h_1$ is an arc. (See Theorem 8.) Hence, by Lemma 1, h_1 is an end element of G_1.

For i equal to 1 or 2, P_i is in the sequential limiting set of a sequence $J_{i1}, J_{i2}, J_{i3}, \cdots$ each term of which is the boundary of a complementary domain of M and is different from J. Since no element of either of these sequences is a subset of H_1^*, and h_1 does not separate M, there exists a positive number δ such that if n is greater than δ, J_{1n} and J_{2n} are subsets of h_1.

Since $J \cdot h_1$ contains an arc, no element of G_2 contains J. Hence, for some m greater than δ, J_{1m} or J_{2m} is a subset of the common part of an element of G_1 and an element of G_2. This involves a contradiction.

Lemma 3. *If M is a W_2' set and K_1 and K_2 are nondegenerate continua such that their common part is the point P and each of them is the sum of some elements*[11] *of $S(M)$, then P is not a limit point of $B(M)-(K_1+K_2)$.*

Proof. Suppose P is a limit point of $B(M)-(K_1+K_2)$. Let S_3 denote a sequence $J_{31}, J_{32}, \cdots$, having P in its sequential limiting set such that for each i, J_{3i} intersects $B(M)-(K_1+K_2)$ and is the boundary of a complementary domain of M. If j is 1 or 2 let S_j denote a sequence $J_{j1}, J_{j2}, \cdots$, having P in its sequential limiting set such that, for each i, J_{ji} lies in K_j and is the boundary of a complementary domain of M.

Let G_1 and G_2 be two collections with respect to which M is a W_2' set. Let g_1 be the element of G_1 containing P. Suppose that a component, U, of $M-g_1$ has the property that each of two of the sequences S_1, S_2, and S_3 has a subsequence each element of which intersects U and has P in its sequential limiting set. One of these two sequences is either S_1 or S_2. Suppose the two sequences are S_1 and S_3. Since K_1 is a continuum and intersects U, there exists a positive integer n, a non-end element g' of G_1, and a boundary, J', of a complementary domain of M such that g' is a subset of U, J' is a subset of K_1, and $J' \cdot g'$ and $J_{3n} \cdot g'$ exist and are totally disconnected. However, it follows from Theorem 6 that this is impossible. Consequently, for one of the sequences S_1, S_2, and S_3—say S_1—there is a positive integer δ_1 such that if n is greater than δ_1, J_{1n} is a subset of g_1. It follows from Lemma 1 that g_1 is an end element of G_1 and consequently that $M-g_1$ has only one component. Hence, for one of the sequences S_2 and S_3—say S_2—there is a positive integer δ_2 such that if n is greater than δ_2, J_{2n} is a subset of g_1.

In a similar manner it may be shown that if g_2 is the element of G_2 containing P, then for one of the sequences S_1 and S_2—say S_1—there is a positive integer δ greater than δ_1 such that if n is greater than δ, J_{1n} is a subset of g_2. Since J_{1n} is also a subset of g_1, this is impossible; hence the supposition that P is a limit point of $B(M)-(K_1+K_2)$ is false.

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(11) It follows from Lemma 1 and Theorem 9 that no nondegenerate continuum of elements of $S(M)$ is made up entirely of elements of $S(M)$ which are ordinary points of M.

LEMMA 4. *If M is a W_2' set, there is no simple closed curve in the space $Q_{S(M)}$.*

Proof. It follows from Lemmas 1 and 2 and Theorem 9 that $S(M)$ is an upper semicontinuous collection of type 2, so the space $Q_{S(M)}$ exists. Suppose K is a simple closed curve in the space $Q_{S(M)}$. Since no continuum in $Q_{S(M)}$ is the sum of points([12]) of $S(M)$ and since M is not separated by any finite point set([13]), it follows that infinitely many elements of K are boundaries of complementary domains of M.

There exists such a boundary, J, and an integer i, less than 3, such that no end element of G_i intersects J. Let h_1 and h_2 denote the end elements of the collection, H, of all elements of G_i that intersect J and let H_1 denote $H-(h_1+h_2)$. The point set $G_i^*-H^*$ is the sum of two mutually separated connected point sets L and N.

Since K is a simple closed curve, it follows from Lemmas 1, 2, and 3 that there exist two points, P_1 and P_2, of J such that $K^*-[J-(P_1+P_2)]$ is a continuum. Call it K'. If P_1 or P_2 were in H_1^*, $K\cdot H_1^*$ would contain the boundary of some complementary domain of M, which we have seen to be impossible. Since $h_1\cdot J$ and $h_2\cdot J$ are connected, they are degenerate. These two points are P_1 and P_2. Hence L and N intersect K' and therefore so does H_1^*. This, however, is impossible. Hence Lemma 4 is proved.

Proof of Theorem 16. First suppose M is a W_2' set. Let Z denote the set of all "points" of the space $Q_{S(M)}$. It follows from Lemmas 3 and 4 that every component of Z is an atriodic dendron and therefore an arc or a point. It follows from Lemma 3 that if the component K of Z is an arc and K^* contains a limit point of $B(M)-K^*$, then that point is a point of an end element of K. By Lemma 2, no end element of K contains more than one such point. By Lemma 3, if an end element of K is a simple closed curve, J, then no point of J is a limit point both of K^*-J and of $B(M)-K^*$. By Lemma 3, if the simple closed curve J is a non-end element of the nondegenerate component K of Z, then J contains only two limit points of K^*-J. Furthermore, if J is any nondegenerate element of $S(M)$, J contains no more than two limit points of $B(M)-J$.

If J, the boundary of a complementary domain of M, contains two limit points of $B(M)-J$, denote these points by P_J and Q_J. If it contains only one limit point of $B(M)-J$, denote it by P_J and let Q_J be a point of J different from P_J. If it contains no limit point of $B(M)-J$, let P_J and Q_J be two points of J. There is an upper semicontinuous collection, H_J, of mutually exclusive point sets filling up J such that (1) H_J is an arc with respect to its elements, (2) the end elements of H_J are P_J and Q_J, and (3) every non-end element of H_J is composed of two points separated in J by P_J+Q_J.

([12]) See footnote 11.

([13]) R. H. Bing, loc. cit., Theorem 1.

Let T_M be the collection consisting of all the elements of H_J for every simple closed curve J that is the boundary of a complementary domain of M and all the points of M which are not contained in an element of H_J for any J. The collection T_M is an upper semicontinuous collection of mutually exclusive point sets which fills up M and is the surface of a sphere with respect to its elements. Let $T_{B(M)}$ denote the collection of all elements of T_M which are subsets of $B(M)$. With respect to its elements regarded as points $T_{B(M)}$ is a closed point set each component of which is either an arc or a point and whose components form a contracting sequence(14). Furthermore, if the nondegenerate component, L, of the set of all elements ("points") of $T_{B(M)}$ contains a limit "point" of $T_{B(M)}-L$, that "point" is an "end point" of L. Therefore $T_{B(M)}$ satisfies the conditions of a theorem(15) of R. L. Moore and J. R. Kline and as a result there is an arc α in T_M which contains $T_{B(M)}$ and which is such that neither of its end elements belongs to $T_{B(M)}$. Each element of α that is not an element of $T_{B(M)}$ is a point of M. Let S' be the collection consisting of $S(M)$ plus the collection of all elements of α not in $T_{B(M)}$. This collection satisfies all the conditions required of S' in the statement of Theorem 16.

Suppose now that M is a continuum satisfying the conditions of Theorem 16. It will be shown that M is a W_2' set. It is easily seen that M is a continuous curve and that it is a connected domain plus its boundary. Let K denote the sum of the elements of the collection S' and let K' denote the sum of the end elements of S' and the cut points of K.

Let M', C_2, G_1, and G_2 be as defined in the proof of Theorem 11. Let β be an arc on C_2 which does not intersect the x-axis and let C be a closed subset of β such that there is a reversibly continuous transformation T of C into K' such that if A, B, and O are points of C and O separates A from B in β, then $T(O)$ separates $T(A)$ from $T(B)$ in K. Let C' be the reflection of C in the x axis.

Let G be an upper semicontinuous collection of mutually exclusive point sets filling up M' such that each element of G is either a point of $M'-(C+C')$ or the common part of $C+C'$ and some vertical line. With respect to its elements regarded as points G is topologically equivalent to M and it follows from arguments similar to those used in the proofs of Theorems 2 and 11 that M is a W_2' set.

In a similar way it may be shown that if n is an integer greater than 1, then Theorem 16 remains true if in its statement "W_2'" is replaced by "W_n'."

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(14) This follows from Lemma and Theorem 9.

(15) R. L. Moore and J. R. Kline, *On the most general closed point set through which it is possible to pass a simple continuous arc*, Ann. of Math. vol. 20 (1919) pp. 218–223.